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# On some generalization of the weighted Strichartz estimates for the wave equation and self-similar solutions to nonlinear wave equations

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## 1 Weighted Strichartz estimates

This note is based on our recent joint work of the same title [7].

Let $w$ be a solution to the following Cauchy problem of the inhomogeneous wave equation with zero data,

$$\partial_t^2 w - \Delta w = F, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n \equiv \mathbf{R}_+^{1+n}, \tag{1.1}$$

$$w|_{t=0} = 0,\ \partial_t w|_{t=0} = 0, \quad x \in \mathbf{R}^n. \tag{1.2}$$

We consider the associated time-space weighted $L^q$-$L^{q'}$ estimates for the solution $w$ of the form

$$\| |t^2 - |x|^2|^a w \|_{L^q(\mathbf{R}_+^{1+n})} \le C \| |t^2 - |x|^2|^b F \|_{L^{q'}(\mathbf{R}_+^{1+n})}, \quad 2 \le q \le \frac{2(n+1)}{n-1}, \tag{1.3}$$

which is called the weighted Strichartz estimates. Here, $q'$ is the conjugate exponent to $q$. We notice that estimates (1.3) are recognized as the hyperbolic version of the following Carleman type estimates

$$\| |x|^{-a} f \|_{L^q(\mathbf{R}^n)} \le \| |x|^{-b} \Delta f \|_{L^{q'}(\mathbf{R}^n)}, \quad 2 \le q \le \frac{2n}{n-2}.$$

See [5], for example.

Estimates (1.3) were proved by Georgiev-Lindblad-Sogge [3] under the following conditions

$$a < \frac{n-1}{2} - \frac{n}{q}, \quad b > \frac{1}{q}, \quad \mathrm{supp} F \subset \{(t,x);\, |x| < t-1\}. \tag{1.4}$$

Using these estimates, they solved part of Strauss' conjecture concerning the existence of time-global solutions to the Cauchy problem of nonlinear wave equation with compactly supported, smooth, small initial data. Later, D'Ancona-Georgiev-Kubo [1] removed the assumption on the support of $F$ in (1.4). Tataru [22] proved (1.3) when

$$a-b+\frac{n+1}{q}=\frac{n-1}{2},\quad b<\frac{1}{q},\quad \mathrm{supp}F\subset\{(t,x);\,|x|<t\},\tag{1.5}$$

where the first one is related to the scale invariance.

The purpose of this note is to show the estimates (1.3) without the support assumption on $F$ in the *scale invariant* case, which have an application to the existence of the self-similar solutions to nonlinear wave equations as we shall see below. Concerning this, in [9, 10], it was shown that the estimates (1.3) hold if $F$ is radial in space variables without the support assumption on $F$. Precisely, it was proved that the estimates (1.3) hold if

$$a-b+\frac{n+1}{q}=\frac{n-1}{2},\quad \frac{n}{q}-\frac{n-1}{2}<b<\frac{1}{q},\quad F(t,x)=\widetilde{F}(t,|x|).\tag{1.6}$$

except the borderline case $q=2(n+1)/(n-1)$. As compared with the condition (1.5) the assumption on the support of $F$ is removed at the cost of the additional lower bound on $b$, namely, $b>n/q-(n-1)/2$.

In this note, we remove the assumption of radial symmetry on $F$ in (1.6):

**Theorem 1.1.** *Let $n\geq 2$. Let $q$, $a$, $b$ satisfy $2<q<2(n+1)/(n-1)$,*

$$a-b+\frac{n+1}{q}=\frac{n-1}{2},\quad \frac{n}{q}-\frac{n-1}{2}<b<\frac{1}{q}.\tag{1.7}$$

*Then, for the solution $w$ to (1.1), (1.2), the estimate*

$$\left\||t^2-|x|^2|^a w\right\|_{L^q_{t,r}L^2_\omega}\leq C\left\||t^2-|x|^2|^b F\right\|_{L^{q'}_{t,r}L^2_\omega}\tag{1.8}$$

*holds.*

Here, for $G=G(t,x)$, the norm $\|\cdot\|_{L^p_{t,r}L^2_\omega}$ is defined by

$$\|G\|_{L^p_{t,r}L^2_\omega}=\left\{\int_0^\infty\int_0^\infty\|G(t,r\cdot)\|^p_{L^2(S^{n-1})}r^{n-1}dr\,dt\right\}^{1/p},\tag{1.9}$$

using polar coordinates $x=r\omega$, $r>0$, $\omega\in S^{n-1}$. Theorem 1.1 says that, the introduction of $L^2$ space on the sphere enables us to remove the assumption of radial symmetry on $F$.

In odd space dimensions, we are able to improve the above result. Namely, we obtain a gain of regularity with respect to angular variables.

**Theorem 1.2.** *Let $n\geq 3$ be odd. Let $q$, $a$, $b$ satisfy $4(n-1)/(2n-3)<q<2(n+1)/(n-1)$,*

$$a-b+\frac{n+1}{q}=\frac{n-1}{2},\quad \frac{n+1}{2q}-\frac{n-1}{4}<b<\frac{1}{q}.\tag{1.10}$$

*Then, for the solution $w$ to (1.1), (1.2),*

$$\left\||t^2-|x|^2|^a w\right\|_{L^q_{t,r}H^{1/2}_\omega}\leq C\left\||t^2-|x|^2|^b F\right\|_{L^{q'}_{t,r}L^2_\omega}\tag{1.11}$$

*holds.*

*Remark* 1.3. The lower bound on $b$ in Theorem 1.2 is strictly greater than the one in Theorem 1.1 for $q > 2$.

Here, $H_\omega^s$ denotes the Sobolev space on $S^{n-1}$ of fractional order $s$ and the norm $\|\cdot\|_{L^q_{t,r}H^s_\omega}$ is defined similarly to (1.9).

The idea of the proof of Theorems 1.1, 1.2 is based on the expansion by spherical harmonics. We derive the expansion of the solution $w$ with respect to spherical harmonics and reduce the estimates essentially to radial case. This idea is due to [13], which treats end point Strichartz estimates for the wave equation in 3 space dimensions using the norm with respect to angular variables. We also notice that similar type of Strichartz estimates are treated in [12].

This note is organized as follows. In Section 2 we prove Theorem 1.1 and give the outline of the proof of 1.2. In Section 3 we give the application of these theorems to the existence of self-similar solutions to nonlinear wave equations.

# 2 Proof of theorems

The proof of Theorem 1.1, 1.2 is based on the expansion of the solution $w$ with respect to the spherical harmonics. We first describe its expansion precisely.

For $k \geq 0$, We denote by $\mathcal{H}_k$ the space of spherical harmonics of degree $k$ on $S^{n-1}$, by $\alpha_k$ its dimension, and by $\{Y_1^k, \cdots, Y_{\alpha_k}^k\}$ the orthonormal basis of $\mathcal{H}_k$. It is well known that $L^2(S^{n-1}) = \oplus_{k=0}^{\infty}\mathcal{H}_k$ and that $F(t,x) = F(t, r\theta)$ has the expansion

$$F(t, r\omega) = \sum_{k=0}^{\infty}\sum_{l=1}^{\alpha_k} F_l^k(t,r)\, Y_l^k(\omega). \tag{2.1}$$

Then, by orthogonality, we observe that $\|F(t,r\cdot)\|_{L^2(S^{n-1})} = (\sum_{k,l}|F_l^k(t,r)|^2)^{1/2}$ and more generally,

$$\|F(t,r\cdot)\|_{H^s(S^{n-1})} = \Big\{\sum_{k,l}(1+k(k+n-2))^s|F_l^k(t,r)|^2\Big\}^{1/2}. \tag{2.2}$$

Note that $(-\Delta_{S^{n-1}})Y^k = k(k+n-2)Y^k$ for $Y^k \in \mathcal{H}_k$, where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on $S^{n-1}$.

In the following, we set

$$W_n(t) = (-\Delta)^{-1/2}\sin[(-\Delta)^{1/2}t],$$

where we specially denote the space dimension $n$ for later use. Then, the solution $w$ to (1.1), (1.2) is given by

$$w(t, r\omega) = \int_0^t [W_n(t-s)F(s,\cdot)](r\omega)\, ds,$$

which is written in terms of (2.1) by

$$
\begin{aligned}
w(t,r\omega) &= \int_0^t \big[W_n(t-s)\{\sum_{k=0}^{\infty}\sum_{l=1}^{\alpha_k} F_l^k(s,\lambda)\,Y_l^k(\theta)\}\big](r\omega)\,ds \\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{\alpha_k}\int_0^t \big[W_n(t-s)\{F_l^k(s,\lambda)\,Y_l^k(\theta)\}\big](r\omega)\,ds. \qquad (2.3)
\end{aligned}
$$

Then, we use the following lemma.

**Lemma 2.1.** *Let $Y^k \in \mathcal{H}_k$. Then, for $f \in C_0^\infty((0,\infty))$,*

$$
W_n(t)\big[f(\lambda)Y^k(\theta)\big](r\omega) = r^k\, W_{n+2k}(t)\big[\lambda^{-k}f(\lambda)\big](r)\,Y^k(\omega). \qquad (2.4)
$$

*Remark* 2.2. We apply Lemma 2.1 to compute (2.3). To prove Theorems 1.1, 1.2 it suffices to show for $F \in C_0^\infty(\mathbf{R}_+^{1+n} \setminus \{|x| = 0\})$. In fact, such space is dense in the weighted Lebesgue spaces in question, and then, for each $t \geq 0$,

$$
F_l^k(t,r) = \int_{S^{n-1}} F(t,r\theta)Y_l^k(\theta)\,d\sigma(\theta) \in C_0^\infty((0,\infty)).
$$

Note that since $F \in C_0^\infty(\mathbf{R}_+^{1+n} \setminus \{|x| = 0\})$, $F_l^k(t,r)$ vanishes when $r$ is sufficiently small.

*Proof of Lemma* 2.1. Since $f \in C_0^\infty((0,\infty))$, the left hand side of (2.4) is a classical solution of the Cauchy problem of the wave equation

$$
\partial_t v - \Delta v = 0, \qquad (2.5)
$$

$$
v(0,x) = 0, \quad \partial_t v(0,x) = f(|x|)Y^k(x/|x|). \qquad (2.6)
$$

Thus, if we show that the right hand side of (2.4)

$$
z(t,r\omega) = r^k\,\tilde{z}(t,r)Y^k(\omega)
$$

is also a classical solution of (2.5), (2.6), where $\tilde{z}(t,r) = W_{n+2k}(t)[\lambda^{-k}f(\lambda)](r)$, then by the uniqueness of classical solutions we obtain (2.4). Obviously, $z$ is regular and satisfies (2.6). Therefore, it suffices to show that $z$ satisfies (2.5), which is observed as follows.

$$
\begin{aligned}
&(\partial_t^2 - \Delta)z \\
&= \Big(\partial_t^2 - \partial_r^2 - \frac{n-1}{r}\partial_r - \frac{1}{r^2}\Delta_{S^{n-1}}\Big)r^k\,\tilde{z}Y^k \\
&= r^k\Big(\partial_t^2\tilde{z} - \partial_r^2\tilde{z} - \frac{n+2k-1}{r}\partial_r\tilde{z} - \frac{k(k+n-2)}{r^2}\tilde{z}\Big)Y^k + r^k\,\tilde{z}\frac{k(k+n-2)}{r^2}Y^k \\
&= r^k\Big(\partial_t^2\tilde{z} - \partial_r^2\tilde{z} - \frac{n+2k-1}{r}\partial_r\tilde{z}\Big)Y^k = 0.
\end{aligned}
$$

This completes the proof of Lemma 2.1. □

Applying Lemma 2.1, we obtain the expansion of $w$,

$$w(t, r\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{\alpha_k} S_k(F_l^k)(t,r)\, Y_l^k(\omega), \tag{2.7}$$

where

$$S_k(G)(t,r) = r^k \int_0^t W_{n+2k}(t-s)\big[\lambda^{-k} G(s,\lambda)\big](r)\, ds. \tag{2.8}$$

Using this expansion, we prove Theorems 1.1, 1.2.

*Proof of Theorem* 1.1. By the expansion (2.7), the crucial point of the proof of Theorem 1.1 is to derive the estimate on the coefficients $S_k(F_l^k)$. But the estimate on $S_k(F_l^k)$ needed for the proof of Theorem 1.1 are derived by a similar argument in [9, 10], where the weighted Strichartz estimates under the assumption of radial symmetry are considered. In particular, the following estimates hold.

**Lemma 2.3.** *Let $n \geq 2$. Let $q$, $a$, and $b$ be as in Theorem* 1.1. *Then, there exists a constant $C > 0$ independent of $k$ such that*

$$\big\| |t^2 - r^2|^a r^{(n-1)/q} S_k(G) \big\|_{L^q_{t,r}} \leq C \big\| |t^2 - r^2|^b r^{(n-1)/q'} G \big\|_{L^{q'}_{t,r}}. \tag{2.9}$$

*Proof.* We first consider the case the space dimension $n$ is odd. From (2.8) and the representation of the radial solution (see for instance [21, Lemma 2.2]), we have

$$S_k(G)(t,r) = r^{-(n-1)/2} \int_0^t \int_{|t-s-r|}^{t-s+r} P_{k+(n-3)/2}(\mu) \lambda^{(n-1)/2} G(s,\lambda)\, d\lambda ds, \tag{2.10}$$

where $P_m$ is the Legendre polynomial of degree $m$ and

$$\mu = \frac{r^2 + \lambda^2 - (t-s)^2}{2r\lambda}. \tag{2.11}$$

Then, from the estimate of the Legendre polynomials

$$|P_m(z)| \leq 1, \quad |z| \leq 1, \ m \geq 0 \tag{2.12}$$

and the fact that $|\mu| \leq 1$ if $\lambda \geq |t-s-r|$, we estimate

$$|S_k(G)(t,r)| \leq r^{-(n-1)/2} \int_0^t \int_{|t-s-r|}^{t-s+r} \lambda^{(n-1)/2} |G(s,\lambda)|\, d\lambda ds. \tag{2.13}$$

Thus, to derive the estimate (2.9) it is sufficient to apply the same argument as in [9, Lemma 3.3]. Note that the right hand side of (2.13) is independent of $k$.

We next consider the case $n$ is even. In this case we need two types of representations and estimates of $S_k(G)(t,r)$ to apply the argument in [10]. From (2.8) and the representation of the radial solution (see for instance [21, Lemma 2.3]), we have

$$\begin{aligned} S_k(G)(t,r) = \frac{2}{\pi} r^{-n/2+1} \int_0^t \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}} \\ \times \Big( \int_{|r-\rho|}^{r+\rho} \frac{T_{k+(n-2)/2}(\widetilde{\mu})}{\sqrt{\lambda^2-(r-\rho)^2}\sqrt{(r+\rho)^2-\lambda^2}} \lambda^{n/2} F(s,\lambda) d\lambda \Big) d\rho\, ds, \end{aligned} \tag{2.14}$$

where $T_m$ is the Tschebysheff polynomial of degree $m$ and $\widetilde{\mu} = (\lambda^2 + r^2 - \rho^2)/2r\lambda$. Since $|T_m(z)| \le 1$ for $|z| \le 1$, $m \ge 0$, and $|\widetilde{\mu}| \le 1$ for $\lambda \ge |r-\rho|$, we obtain the pointwise estimate of $S_k(G)(t,r)$ independent of $k$. Similarly, from (2.8) and the representation of radial solution (see for instance [11, Theorem 3.4]), we have

$$\begin{aligned} S_k(G)(t,r) &= r^{-k-n+2} \int_0^t \int_{|t-s-r|}^{t-s+r} \lambda^{k+n-1} K_{k+(n-2)/2}(\lambda, r, t-s) F(s,\lambda)\, d\lambda\, ds \\ &+ r^{-k-n+2} \int_0^{\max(t-r,0)} \int_0^{t-s-r} \lambda^{k+n-1} \widetilde{K}_{k+(n-2)/2}(\lambda, r, t-s) F(s,\lambda)\, d\lambda\, ds. \end{aligned}$$

Here the kernels have the estimates (see [11, Lemma 4.2], [10, Lemma 3.1])

$$\begin{aligned} r^{-k}\lambda^k |K_{k+(n-2)/2}(\lambda,r,\tau)| &\le C\, r^{(n-3)/2} \lambda^{-(n-1)/2} \min(r^{1/2}, \lambda^{1/2})\, (\lambda - \tau + r)^{-1/2}, \\ &\qquad |\tau - r| < \lambda < \tau + r, \\ r^{-k}\lambda^k |\widetilde{K}_{k+(n-2)/2}(\lambda,r,\tau)| &\le C\, r^{(n-3)/2+\sigma} (\tau - r)^{-(n-2)/2-\sigma} (\tau - r - \lambda)^{-1/2}, \\ &\qquad 0 < \lambda < \tau - r, \quad 0 \le \sigma \le 1/2, \end{aligned}$$

where the constants are independent of $k$. These representations and estimates of $S_k(G)(t,r)$ enable us to apply the argument in [10] to derive the estimate (2.9). □

Then, the estimate (1.8) is obtained as follows. By the expansion (2.7) and Lemma 2.3, we have

$$\begin{aligned} \big\| |t^2 - |x|^2|^a w \big\|_{L^q_{t,r} L^2_\omega} &= \big\| |t^2 - r^2|^a r^{(n-1)/q} \big( \sum_{k,l} |S_k(F_l^k)|^2 \big)^{1/2} \big\|_{L^q_{t,r}} \\ &\le \big( \sum_{k,l} \big\| |t^2 - r^2|^a r^{(n-1)/q} S_k(F_l^k) \big\|^2_{L^q_{t,r}} \big)^{1/2} \\ &\le C \big( \sum_{k,l} \big\| |t^2 - r^2|^b r^{(n-1)/q'} F_l^k \big\|^2_{L^{q'}_{t,r}} \big)^{1/2} \\ &\le C \big\| |t^2 - r^2|^b r^{(n-1)/q'} \big( \sum_{k,l} |F_l^k|^2 \big)^{1/2} \big\|_{L^{q'}_{t,r}} \\ &\le C \big\| |t^2 - |x|^2|^b F \big\|_{L^{q'}_{t,r} L^2_\omega}, \end{aligned}$$

where we have used Minkowski's integral inequality repeatedly, since $q > 2$ and $q' < 2$. □

*Proof of Theorem* 1.2. For the proof of Theorem 1.2, we need improved estimates on $S_k(F_l^k)$ instead of those in Lemma 2.4 and such estimates are derived at least in odd space dimensions.

**Lemma 2.4.** *Let $n \geq 3$ be odd. Let $q$, $a$, and $b$ be as in Theorem 1.2. Then, there exists a constant $C > 0$ independent of $k \geq 1$ such that*

$$\big\| |t^2 - r^2|^a r^{(n-1)/q} S_k(G) \big\|_{L^q_{t,r}} \leq C k^{-1/2} \big\| |t^2 - r^2|^b r^{(n-1)/q'} G \big\|_{L^{q'}_{t,r}}. \tag{2.15}$$

*Outline of the proof of Lemma* 2.4. We recall that $S_k(G)$ is given by (2.10) in odd space dimensions. To derive the estimate (2.15) we use another estimate of the Legendre polynomials instead of (2.12). Namely,

$$|P_m(z)| \leq C m^{-1/2} (1 - |z|^2)^{-1/4}, \quad |z| < 1, \ m \geq 1. \tag{2.16}$$

(See [2, §1.6].) Then, from (2.10), we have

$$\begin{aligned} |S_k(G)(t,r)| \leq{}& C k^{-1/2} r^{-(n-1)/2} \\ &\times \int_0^t \int_{|t-s-r|}^{t-s+r} (1-\mu^2)^{-1/4} \lambda^{(n-1)/2} |G(s,\lambda)| \, d\lambda ds. \end{aligned} \tag{2.17}$$

Note that $\mu$ is given by (2.11), and thus

$$\begin{aligned} &(1-\mu^2)^{-1/4} \\ =& \frac{\sqrt{2}\, r^{1/2} \lambda^{1/2}}{(r+\lambda+t-s)^{1/4} (r+\lambda-t+s)^{1/4} (t-s+r-\lambda)^{1/4} (t-s-r+\lambda)^{1/4}}. \end{aligned}$$

We observe that the estimate of $S_k(G)(t,r)$ in this case is similar to that of even space dimensions (see (2.14)). In fact, applying the similar method in [10], which treats the weighted Strichartz estimates in even space dimensions, we are able to reduce the estimate (2.15) to the following weighted Hardy-Littlewood-Sobolev inequality.

**Lemma 2.5** ([19]). *Let $0 < \lambda < n$, $1 < r, s < \infty$. Let $\alpha < n/s'$ and $\beta < n/r'$ with $\alpha + \beta \geq 0$ satisfy $1/s + 1/r + (\lambda + \alpha + \beta)/n = 2$. Then,*

$$\left| \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{f(x) g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} dx\, dy \right| \leq C \|f\|_{L^s(\mathbf{R}^n)} \|g\|_{L^r(\mathbf{R}^n)}.$$

□

Using Lemma 2.4, estimates (1.11) are obtained as follows. By the expansion (2.7) and (2.2), we have

$$\begin{aligned}\big\||t^2-|x|^2|^a w\big\|_{L^q_{t,r}H^{1/2}_\omega} &= \Big\||t^2-r^2|^a r^{(n-1)/q}\Big\{\sum_{k,l}(1+k(k+n-2))^{1/2}|S_k(F^k_l)|^2\Big\}^{1/2}\Big\|_{L^q_{t,r}} \\ &\le C\Big(\sum_{k,l}(1+k)\big\||t^2-r^2|^a r^{(n-1)/q}S_k(F^k_l)\big\|^2_{L^q_{t,r}}\Big)^{1/2} \\ &\le C\Big(\sum_{k,l}\big\||t^2-r^2|^b r^{(n-1)/q'}F^k_l\big\|^2_{L^{q'}_{t,r}}\Big)^{1/2} \\ &\le C\,\Big\||t^2-r^2|^b r^{(n-1)/q'}\Big(\sum_{k,l}|F^k_l|^2\Big)^{1/2}\Big\|_{L^{q'}_{t,r}} \\ &\le C\,\big\||t^2-|x|^2|^b F\big\|_{L^{q'}_{t,r}L^2_\omega},\end{aligned}$$

where we have used Minkowski's integral inequality repeatedly, since $q > 2$ and $q' < 2$. □

# 3 Existence of self-similar solutions

As an application of Theorems 1.1, 1.2, we are able to show the existence of self-similar solutions to the nonlinear wave equation

$$\partial_t^2 u - \Delta u = |u|^p, \quad (t,x) \in (0,\infty)\times \mathbf{R}^n. \tag{3.1}$$

The solution $u$ to (3.1) is called a *self-similar solution* if $u$ satisfies

$$u(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x) \tag{3.2}$$

for all $\lambda > 0$. Letting $\lambda = 1/t$, $u(1,\cdot) = W(\cdot)$, we observe that self-similar solutions are solutions of the following form

$$u(t,x) = t^{-\frac{2}{p-1}} W(x/t).$$

From such scaling properties, it is known that self-similar solutions are useful to investigate the asymptotic behavior of the time-global solutions as $t \to \infty$ (See [14], for example).

It is known that there is a close connection between the existence of the self-similar solutions to (3.1) and the power $p$ of the nonlinear term. In fact, in three space dimensions, Pecher [17] proved that if $p > 1+\sqrt{2}$, there exist self-similar solutions, and if $p \le 1+\sqrt{2}$, self-similar solutions do not exist. We intended to extend such sharp existence results of self-similar solutions to higher dimensions. We denote by $p_0(n)$ the positive root of

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Then, $p_0(3) = 1+\sqrt{2}$ and we expect $p_0(n)$ to be the critical power concerning the existence of self-similar solutions to the equation (3.1). We notice that $p_0(n)$ is the critical exponent concerning the existence of time-global solutions to the Cauchy problem of the equation (3.1) with compactly supported, small, smooth initial data. (See John [6], Georgiev-Lindblad-Sogge [3] and references therein.) So, it is natural to expect $p_0(n)$ to be the one because self-similar solutions are also time-global solutions.

Concerning this problem, in 2 and 3 space dimensions, Hidano [4] proved the existence of self-similar solutions when $p > p_0(n)$. In [9, 10] the first and the third author proved the existence of radially symmetric self-similar solutions when $p > p_0(n)$ for $n \geq 2$.

*Remark* 3.1. Precisely, the above results show the existence of self-similar solutions when $p_0(n) < p < (n+3)/(n-1)$. The existence of self-similar solutions for large $p$ was treated in [16, 18].

As an application of Theorems 1.1, 1.2, we have the following result.

**Theorem 3.2.** *Let $2 \leq n \leq 5$. For any $p$ with $p_0(n) < p < (n+3)/(n-1)$, let $\phi$, $\psi \in C^\infty(\mathbf{R}^n \setminus \{0\})$ be homogeneous of degree $-2/(p-1)$, $-2/(p-1)-1$, respectively. Then, if $\varepsilon > 0$ is sufficiently small, there exists a unique time-global solution $u$ to* (3.1) *with*

$$u(0,x) = \varepsilon\phi(x), \quad \partial_t u(0,x) = \varepsilon\psi(x) \tag{3.3}$$

*satisfying*

$$\left\| |t^2 - |x|^2|^\gamma u; \mathcal{L}^{p+1}_{t,r} H^{(n-1)/2+\delta}_\omega \right\| \leq C\varepsilon,$$

*where $\gamma = 1/(p-1) - (n+1)/2(p+1)$ and $\delta > 0$ sufficiently small.*

Here, $\mathcal{L}^q_{t,r}$ denotes the weak Lebesgue spaces on $\mathbf{R}_+ \times \mathbf{R}_+$ and $\|\cdot\|_{\mathcal{L}^q_{t,r} H^s_\omega}$ is defined by

$$\|G\|_{\mathcal{L}^q_{t,r} H^s_\omega} \equiv \sup_{\lambda > 0} \lambda \left| \{(t,r);\ \|G(t,r\cdot)\|_{H^s(S^{n-1})} > \lambda\} \right|^{1/p}.$$

*Remark* 3.3. By the homogeneity of the data (3.3) and the uniqueness of solutions, the solution obtained in Theorem 3.2 is to be self-similar. That is, self-similar solutions to (3.1) are shown to exist when $2 \leq n \leq 5$, $p > p_0(n)$.

*Remark* 3.4. Sobolev type embedding theorem on the unit sphere

$$H^s(S^{n-1}) \hookrightarrow L^\infty(S^{n-1}) \quad \text{for } s > \frac{n-1}{2} \tag{3.4}$$

is basic to our estimates on the nonlinear term, which in turn causes the restriction $n \leq 5$.

The proof of Theorem 3.2 is essentially the same as that of [9, Theorem 1.1], which shows the existence of radially symmetric self-similar solutions by the standard fixed point arguments using weighted Strichartz estimates of radial case. In fact, we can translate those proofs into $H^{(n-1)/2+\delta}(S^{n-1})$-valued functions. For example, we have the following theorem interpolating the estimates in Theorems 1.1 and 1.2, respectively.

**Theorem 3.5.** *Let $n \geq 2$ and let $s \geq 0$. For $2 < q < 2(n+1)/(n-1)$ and $(n-1)/q < \alpha < (n-1)/q'$, we set*

$$a = \frac{\alpha}{2} - \frac{n+1}{2q}, \quad b = \frac{\alpha}{2} + \frac{n+1}{2q} - \frac{n-1}{2}. \tag{3.5}$$

*Then, there exists a constant $C > 0$ such that for any function $F$ which is homogeneous of degree $-\alpha - 2$, i.e.*

$$F(\lambda t, \lambda x) = \lambda^{-\alpha-2} F(t, x), \quad (t, x) \in \mathbf{R}_+^{1+n}, \ \lambda > 0,$$

*we have*

$$\big\| |t^2 - |x|^2|^a w \big\|_{\mathcal{L}^q_{t,r} H^s_\omega} \leq C \big\| |t^2 - |x|^2|^b F \big\|_{\mathcal{L}^{q'}_{t,r} H^s_\omega}. \tag{3.6}$$

**Theorem 3.6.** *Let $n \geq 3$ be odd and let $s \geq 1/2$. For $4(n-1)/(2n-3) < q < 2(n+1)/(n-1)$ and $(n-1)/2 < \alpha < (n-1)/q'$, we set $a$ and $b$ as in (3.5). Then, there exists a constant $C > 0$ such that for any function $F$ which is homogeneous of degree $-\alpha - 2$, we have*

$$\big\| |t^2 - |x|^2|^a w \big\|_{\mathcal{L}^q_{t,r} H^s_\omega} \leq C \big\| |t^2 - |x|^2|^b F \big\|_{\mathcal{L}^{q'}_{t,r} H^{s-1/2}_\omega}. \tag{3.7}$$

So, we omit the proof of Theorem 3.2 here without illustrating the different point, that is, the estimate of the nonlinear term of the equation (3.1) in terms of $H^{(n-1)/2+\delta}(S^{n-1})$-norm. For that estimate we use the following proposition.

**Proposition 3.7.** *Let $n \geq 2$ and let $p > 1$. For $s \geq s_0$, we assume $p > s_0$ and $s > (n-1)/2$. Then,*

$$\| |g|^p \|_{H^{s_0}(S^{n-1})} \leq C \|g\|^p_{H^s(S^{n-1})}, \tag{3.8}$$

$$\| |g|^p - |h|^p \|_{L^2(S^{n-1})} \leq C(\|g\|^{p-1}_{H^s(S^{n-1})} + \|h\|^{p-1}_{H^s(S^{n-1})}) \|g - h\|_{L^2(S^{n-1})}. \tag{3.9}$$

*Proof.* The estimate (3.8) follows from the Moser type estimate

$$\| |g|^p \|_{H^{s_0}(S^{n-1})} \leq C \|g\|^{p-1}_{L^\infty(S^{n-1})} \|g\|_{H^{s_0}(S^{n-1})}$$

with $p > \max(s_0, 1)$ and the Sobolev embedding (3.4). The estimate (3.9) follows from the Hölder inequality and also the Sobolev embedding (3.4). □

Combining the above proposition with $s_0 = s$ with (3.6), we are able to estimate the nonlinear term when $2 \leq n \leq 4$. In fact,

$$p_0(n) > \frac{n-1}{2} \quad \text{for } 2 \leq n \leq 4$$

and thus

$$\||u|^p\|_{H^s(S^{n-1})} \le C\|u\|^p_{H^s(S^{n-1})}$$

holds if $p > p_0(n)$, $p > s > (n-1)/2$. When $n = 5$, we use (3.7) and Proposition 3.7 with $s_0 = s - 1/2$. In fact,

$$p_0(5) > \left(\frac{n-1}{2} - \frac{1}{2}\right) = \frac{3}{2} \quad \text{for } n = 5,$$

and thus

$$\||u|^p\|_{H^{s-1/2}(S^4)} \le C\|u\|^p_{H^s(S^4)}$$

holds if $p > p_0(5)$, $p > s - 1/2 > 3/2$. Thus, a gain of regularity in (3.7) is used effectively.

*Remark* 3.8. $p_0(n)$ is monotone decreasing as $n \to \infty$, which goes to 1. Note that $p_0(4) = 2$ and $p_0(5) = (3 + \sqrt{17})/4$ ($\fallingdotseq 1.75$).

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